Maths Challenge No. 5

7 November, 2022

Question 1. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are non-constant, differentiable functions, with

$$f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x+y) = f(x)g(y) + g(x)f(y),$$

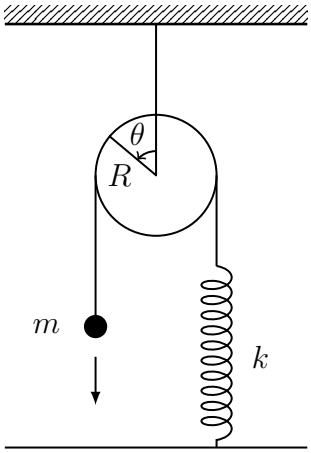
for all $x, y \in \mathbb{R}$. Show that, if $f'(0) = 0$, then $(f(x))^2 + (g(x))^2 = 1$ for all x.

Question 2. Consider a square, with side length s. Find (possibly in terms of s) the average distance from one of the vertices of the square to the other points in the square.

The following results might be helpful

$$\int_0^a \sec^3 x \, \mathrm{d}x = \sec a \tan a - \ln|\sec a + \tan a|, \quad \text{and} \quad \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

Question 3. A block of mass m is attached to one end of a string that passes over a pulley. The radius of the pulley is R, and its moment of inertia is I. There is no friction at its axle and the string does not slip over the pulley. The other end of this string is attached to a spring, with spring constant k, that is fixed to the ground. The pulley hangs from the ceiling through another string. Both strings are massless and inextensible. (See figure.)
The system is initially at rest at its equilibrium position with $\theta = \theta_0$. The block is given a small push downwards, and it begins to oscillate vertically about its equilibrium position, with angular frequency ω.
Let T and V denote the total kinetic and potential energies, respectively.



(i) Calculate T and V in terms of θ, θ_0, and other parameters of the system.

(ii) Hence show that

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \frac{kR^2\varphi}{mR^2 + I} = 0,$$

with $\varphi \coloneqq \theta - \theta_0$. [This is the angle as measured from the equilibrium position.]

(iii) Calculate the time period, T, of the oscillations.

[Note: The aim of this question was to analyse a mechanical system using energy considerations, rather than forces. A refined version of this approach is known as the *Lagrangian formulation* of the classical mechanics.]